Taylor expansion and error

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Taylor’s theorem

**Theorem.** Under suitable conditions,

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \]
\[
+ \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x),
\]

with

\[
R_n(x) = \frac{f^{(n+1)}(x^*)}{(n + 1)!}(x - x_0)^{n+1}
\]

for some \( x^* \in (x_0, x) \).

\( R_n(x) \) is an error term. Under what conditions is \( R_n(x) \) small?

**Note.** There are many alternative statements of Taylor’s theorem.
Example. \( f(x) = \sin(x), \ x_0 = \frac{\pi}{4} \).

Zeroth order Taylor series (Figure 1):

\[ p_0(x) = f(x_0) = \frac{1}{\sqrt{2}} \]

First order Taylor series (Figure 2):

\[ p_1(x) = f(x_0) + f'(x_0)(x - x_0) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \frac{\pi}{4}) \]

Second order Taylor series (Figure 3):

\[ p_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \]

\[ = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \frac{\pi}{4}) - \frac{1}{2\sqrt{2}}(x - \frac{\pi}{4})^2 \]
Example. $f(x) = \sin(x)$, $x_0 = \frac{\pi}{4}$.

Zeroth order Taylor expansion:

$$f(x) = f(x_0) + R_0(x)$$
$$\sin(x) = \frac{1}{\sqrt{2}} + R_0(x)$$
$$R_0(x) = \sin(x) - \frac{1}{\sqrt{2}}$$

Figure 5 is a plot of $\frac{R_0(x)}{x - \frac{\pi}{4}}$. It shows that

$$\frac{R_0(x)}{x - \frac{\pi}{4}} \approx \frac{1}{\sqrt{2}} \text{ for } x \text{ near } \frac{\pi}{4},$$

i.e.,

$$R_0(x) \approx \frac{1}{\sqrt{2}}(x - \frac{\pi}{4}) \text{ for } x \text{ near } \frac{\pi}{4}.$$
Example. \( f(x) = \sin(x) \), \( x_0 = \frac{\pi}{4} \).

First order Taylor expansion:

\[
\begin{align*}
    f(x) &= f(x_0) + f'(x_0)(x - x_0) + R_1(x) \\
    \sin(x) &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x - \frac{\pi}{4}) + R_1(x) \\
    R_1(x) &= \sin(x) - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(x - \frac{\pi}{4})
\end{align*}
\]

Figure 6 is a plot of \( \frac{R_1(x)}{(x - \pi/4)^2} \). It shows that

\[
\frac{R_1(x)}{(x - \pi/4)^2} \approx -\frac{1}{2\sqrt{2}} \quad \text{for } x \text{ near } \frac{\pi}{4},
\]

i.e.,

\[
R_1(x) \approx -\frac{1}{2\sqrt{2}}(x - \frac{\pi}{4})^2 \quad \text{for } x \text{ near } \frac{\pi}{4}.
\]
Example

Find a Taylor expansion for $\log x$ about $x_0$. 
Absolute and relative error

Suppose that \( \tilde{x} \) is an approximation for \( x \).

The \textit{absolute error} is

\[ |x - \tilde{x}|. \]

The \textit{relative error} is

\[ \frac{|x - \tilde{x}|}{|x|}. \]
Absolute and relative error

Relative error is nearly synonymous with “digits of precision.”

*Example.* Suppose $x = 3513.6306$ and $\tilde{x} = 3513.6786$.

- $\tilde{x}$ is accurate to about 5 digits.
- The relative error is

$$\frac{|x - \tilde{x}|}{|x|} = \frac{0.048}{3513.6306} = 1.4 \ldots \times 10^{-5}.$$  

We will prefer the concept of *relative error* over the language of “digits,” because “digits” are surprisingly tricky. (To me, anyway.)