Rational Trigonometry

There are an infinite number of points on the unit circle whose coordinates are each rational numbers. Indeed, every Pythagorean triple gives one!

Consider the tetrahedron with vertices (1, 1, 1), (1, −1, −1), (−1, 1, −1), (−1, −1, 1). What is the cosine of the angle formed by rays from the origin to two of these vertices?

The cosine is $-\frac{1}{3}$, and the sine is $\frac{2\sqrt{2}}{3}$. 
Rational Trigonometry

A rational angle is an angle whose degree measure is a rational number. Equivalently, it's an angle whose radian measure is a rational number times π.

It's not too hard to see that the sine and cosine of a rational angle are algebraic numbers.

Toward this end, one may use the angle sum formulas for sine and cosine to build up expressions for cos(3a), cos(4a), cos(5a), etc. that are polynomials in cos(a).

\[
\begin{align*}
\cos(x + y) &= \cos(x) \cos(y) - \sin(x) \sin(y) \\
\sin(x + y) &= \sin(x) \cos(y) + \cos(x) \sin(y) \\
\cos(3a) &= \cos(2a + a) \\
&= \cos(2a) \cos(a) - \sin(2a) \sin(a) \\
&= (2 \cos^2(a) - 1) \cos(a) - (2 \sin(a) \cos(a)) \sin(a) \\
&= 2 \cos^3(a) - \cos(a) - 2 \sin^2(a) \cos(a) \\
&= 2 \cos^3(a) - \cos(a) - 2(1 - \cos^2(a)) \cos(a) \\
&= 4 \cos^3(a) - 3 \cos(a)
\end{align*}
\]

Rational Trigonometry

This process builds the Chebyshev polynomials of the first kind.

<table>
<thead>
<tr>
<th>(a)</th>
<th>(\cos(2a))</th>
<th>(\cos(3a))</th>
<th>(\cos(4a))</th>
<th>(\cos(5a))</th>
<th>(\cos(6a))</th>
<th>(\cos(7a))</th>
<th>(\cos(8a))</th>
<th>(\cos(9a))</th>
<th>(\cos(10a))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2a</td>
<td>(2 \cos^2(a) - 1)</td>
<td>(4 \cos^3(a) - 3 \cos(a))</td>
<td>(8 \cos^4(a) - 8 \cos^2(a) + 1)</td>
<td>(16 \cos^5(a) - 20 \cos^3(a) + 5 \cos(a))</td>
<td>(32 \cos^6(a) - 48 \cos^4(a) + 18 \cos^2(a) - 1)</td>
<td>(64 \cos^7(a) - 112 \cos^5(a) + 56 \cos^3(a) - 7 \cos(a))</td>
<td>(128 \cos^8(a) - 256 \cos^6(a) + 160 \cos^4(a) - 32 \cos^2(a) + 1)</td>
<td>(256 \cos^9(a) - 576 \cos^7(a) + 432 \cos^5(a) - 120 \cos^3(a) + 9 \cos(a))</td>
<td>(512 \cos^{10}(a) - 1280 \cos^8(a) + 1120 \cos^6(a) - 400 \cos^4(a) + 50 \cos^2(a) - 1)</td>
</tr>
</tbody>
</table>
Rational Trigonometry

Letting $a = \frac{m}{n} \pi$, we see that $\cos(na) = \pm 1$, so $\cos(a)$ is a root of the appropriate Chebyshev polynomial plus or minus 1.

**Example:** With $n = 3$, we get $\cos(\pi/3)$ is a root of $4x^3 - 3x + 1$, so the cosine of $\pi/3$ is algebraic. (But we knew that!)

This reasoning shows $\cos(\frac{m}{n} \pi)$ is algebraic for all rational numbers $\frac{m}{n}$.

Finally, $\sqrt{1 - \cos^2(\frac{m}{n} \pi)}$ is also algebraic, so $\sin(\frac{m}{n} \pi)$ is algebraic too.

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Rational Trigonometry

Which rational angles have a *super-easy-to-remember* sine and cosine? Specifically: Which rational angles $\theta$ have $\sin^2(\theta)$ and $\cos^2(\theta)$ both rational? Here are a few such rational angles in the first quadrant: $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$.

Are there more?

\[
\left(\frac{7}{25}\right)^2 + \left(\frac{24}{25}\right)^2 = 1
\]
\[
\left(-\frac{1}{3}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2 = 1
\]
\[
\left(\frac{\sqrt{2}}{10}\right)^2 + \left(\frac{7\sqrt{2}}{10}\right)^2 = 1
\]
**Rational Trigonometry**

**LEMMA 1.** Let $b_0$ be any rational number with $-2 \leq b_0 \leq 2$. Define a sequence recursively by $b_{k+1} = b_k^2 - 2$. Then:

(i) Each $b_k$ is rational.
(ii) Each $b_k$ satisfies $-2 \leq b_k \leq 2$.
(iii) If the same value is ever attained twice, say $b_m = b_{m+k}$, then $b_m \in \{-2, -1, 0, 1, 2\}$.
(iv) If $b_m \in \{-2, -1, 0, 1, 2\}$ and $m > 0$, then $b_{m-1} \in \{-2, -1, 0, 1, 2\}$.

Thus if the sequence $b_0, b_1, b_2, \ldots$ ever repeats a term, then $b_0 \in \{-2, -1, 0, 1, 2\}$.

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**Rational Trigonometry**

(iii) If the same value is ever attained twice, say $b_m = b_{m+k}$, then $b_m \in \{-2, -1, 0, 1, 2\}$.

*Proof of (iii)*

Let $f(x) = x^2 - 2$, and for any natural number $n$, let $f^n(x) = f(f(\cdots f(x) \cdots ))$ be the $n$-fold composition of $f$. This is a monic polynomial of degree $2^n$, with integer coefficients.

Note that $f^k(b_m) = f^k(f^m(b_0)) = f^{m+k}(b_0) = b_{m+k} = b_m$.

This means $b_m$ is a rational root of $f^k(x) - x$.

This means that $b_m$ is an integer! (Gauss' Lemma: Every rational root of a monic polynomial with integer coefficients must be an integer.)
Rational Trigonometry

**LEMMA 2.**
Let \( \theta \in \mathbb{R} \), and let \( b_0 = 2 \cos(\theta) \). Then \( b_k = 2 \cos(2^k \theta) \) for all \( k \).

**Proof:**
This is a simple inductive argument that makes use of the double angle formula for cosine.

\[
\begin{align*}
  b_k &= b_{k-1}^2 - 2 \\
  &= (2 \cos(2^{k-1} \theta))^2 - 2 \\
  &= 4 \cos^2(2^{k-1} \theta) - 2 \\
  &= 2 (2 \cos(2^{k-1} \theta) - 1) \\
  &= 2 \cos(2^k \theta)
\end{align*}
\]

**THEOREM**
Suppose \( \theta \) is a rational angle, and that \( \cos^2(\theta) \) is rational. Then \( \cos(\theta) \) and \( \sin(\theta) \) both belong to the set \( \{ 0, \pm 1, \pm \frac{1}{2}, \pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{3}}{2} \} \).

**Proof:**
Write \( \theta = \frac{m}{n} \pi \), and we suppose that \( \cos^2(\theta) \) is rational (which implies \( \sin^2(\theta) \) is rational). It suffices to show \( \cos(\theta) \) belongs to the set \( \{ 0, \pm \frac{1}{2}, \pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{3}}{2}, \pm 1 \} \).

Let \( b_0 = 2 \cos(2 \theta) \). By the double angle formula, \( b_0 = 4 \cos^2(\theta) - 2 \), which is rational, and clearly \(-2 \leq b_0 \leq 2\), so this defines the sequence \( b_0, b_1, b_2, \ldots \) of lemma 1.

By lemma 2, \( b_k = 2 \cos\left(\frac{m2^k}{n} \pi\right) \). The key is that the only possible remainders upon division of an integer by \( n \) are 0, 1, 2, \ldots, \( n - 1 \). So after \( n+1 \) terms in the sequence, there must be two terms that are equal. But lemma 1 then tells us that \( b_0 \in \{-2, -1, 0, 1, 2\} \).

This gives \( \cos\left(\frac{m}{n} 2 \pi\right) \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\} \). Or (using the double angle formula yet again):
\[
2 \cos^2\left(\frac{m}{n} \pi\right) - 1 \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}
\]

A few short calculations should allow you to reach the conclusion.
Rational Trigonometry

**COROLLARY**
Suppose $\theta$ is a rational angle, and that $\cos(\theta)$ is rational. Then $\cos(\theta)$ belongs to the set $\{0, \pm 1, \pm \frac{1}{2}\}$.

*Proof:*
Since $\cos^2(\theta)$ is also rational, the first theorem tells us that $\cos(\theta) \in \{0, \pm \frac{1}{2}, \pm \frac{\sqrt{2}}{2}, \pm \frac{\sqrt{3}}{2}, \pm 1\}$. The only rational members of this set are $\{0, \pm 1, \pm \frac{1}{2}\}$.

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Further Reading


Jahnel, Jörg, “When is the (co)sine of a rational angle equal to a rational number?,” arXiv:1006.2938 [math.HO].