# Tossing a coin: It's harder than it looks

Based on Daniel W. Stroock's
"Doing Analysis by Tossing a Coin"

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# Daniel W. Stroock

[picture]

# Theme of the talk

Absolutely bizarre functions arise *inevitably*[1].

[1] Stroock's word

# Our first bizarre example: Brownian motion

# Brownian motion

[2-d animation: animations(1)]

[1-d animation: animations(2)]

# Brownian motion

[money.cnn.com, Get Quotes, Advanced Charting]

# Brownian motion

1827 Robert Brown observes pollen particles.

1900 In his PhD thesis, Bachelier

- provides a partial mathematical model for Brownian motion,
- applies the model to financial markets.

1905 In one year, Einstein publishes on

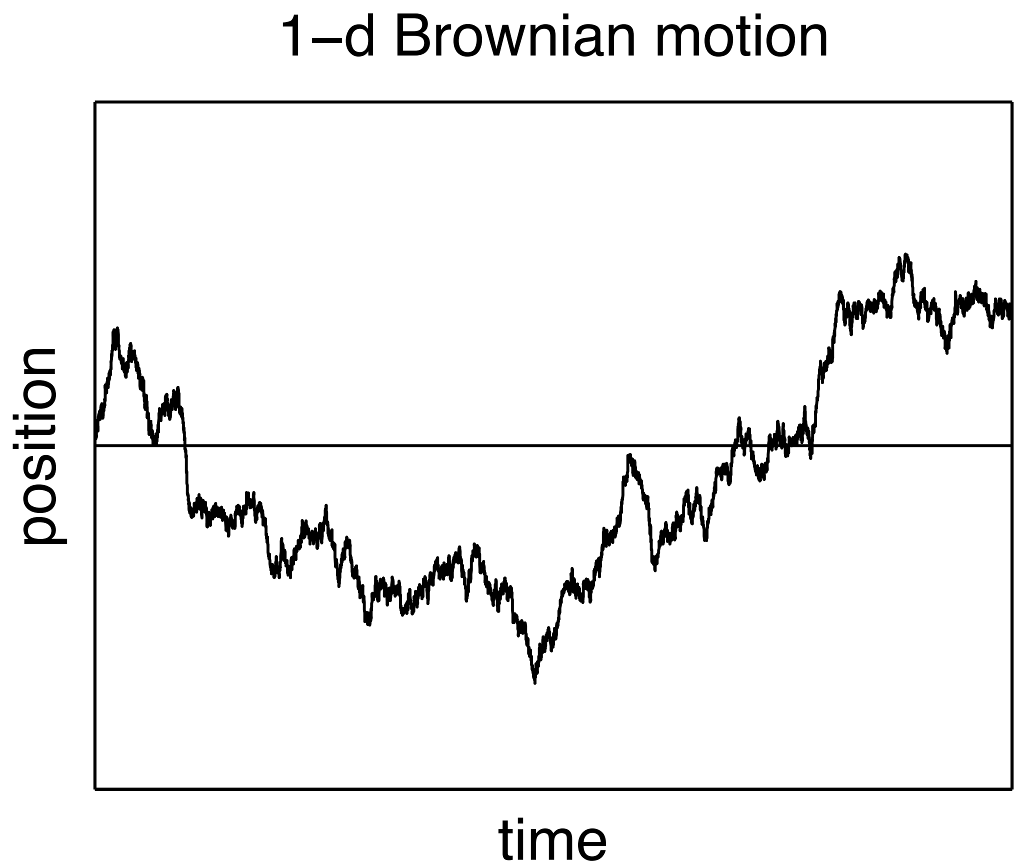
- special relativity,
- the photoelectric effect (quantum physics),
- Brownian motion.

1923 Wiener proves that Brownian motion exists.

1973 Black and Scholes model the stock market with (geometric) Brownian motion. Scholes and colleague Merton win a Nobel prize.

# Brownian motion

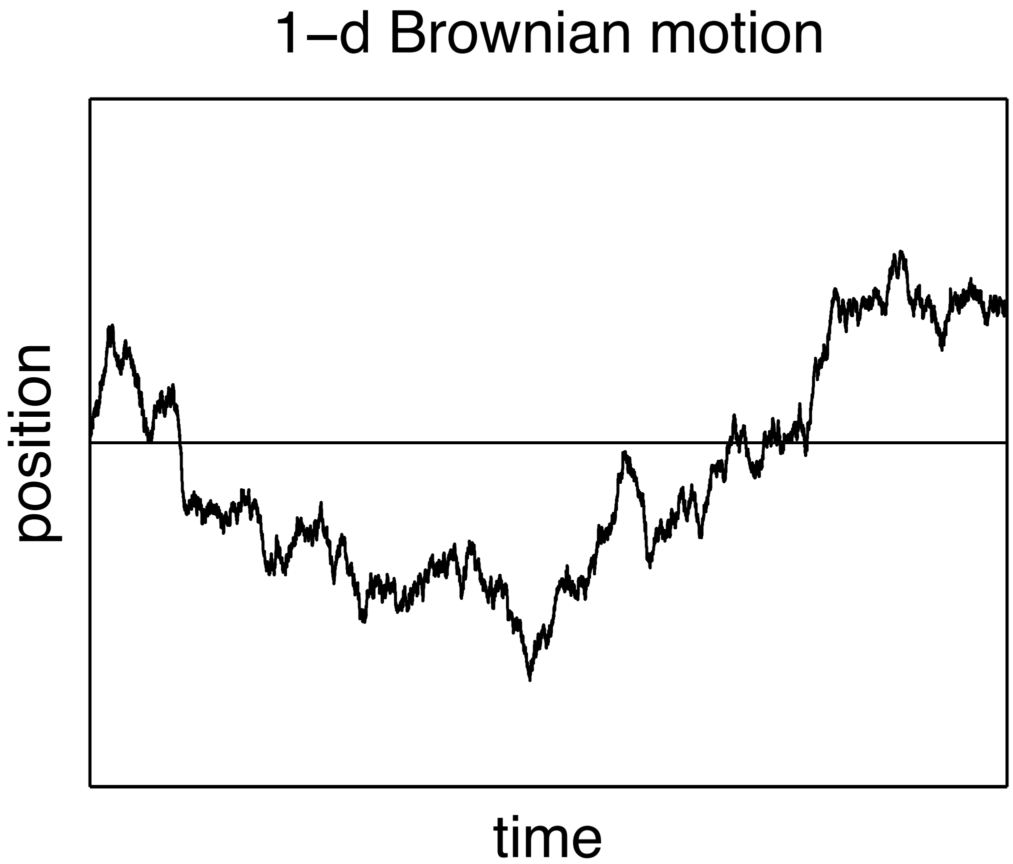
Brownian motion largely defies our tools from calculus.




With probability one, a Brownian path is

- continuous
- nowhere differentiable.

# Brownian motion




As $h \to 0$ from the right, the difference quotient

$$\frac{f(t+h) - f(t)}{h}$$

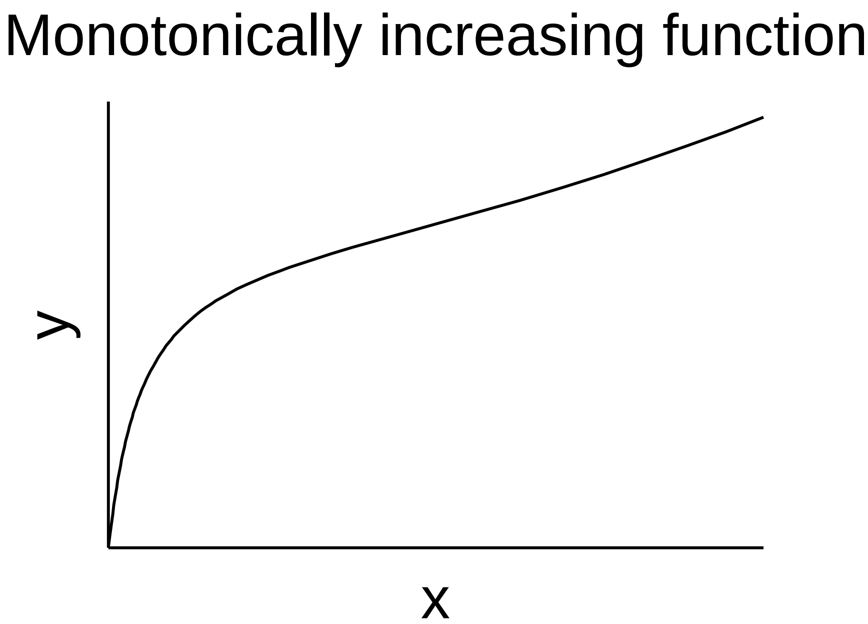
oscillates between $-\infty$ and $+\infty$. Talk about *nondifferentiable*!

Our second bizarre example:

Stroock's "exotic" function

# Is there hope?

Okay, Brownian motion is super weird.
At least we can still believe in *monotone* functions, right?




"At some point in their education, . . . mathematicians learn that every monotone function is *almost everywhere* . . . differentiable."
—DWS

Monotone functions can't possibly be as strange as Brownian motion, ***right?!***

# Our hope is dashed!

[http://homepage.mac.com/barthold.van.acker/realbasic/]

# Stroock's "exotic" function

A function with the following properties.

- non-constant
- continuous
- monotone
- derivative equals 0 *almost everywhere*
- starts at $(0, 0)$, ends at $(1, 1)$, arclength 2

Stroock's function is so strange, we can hardly draw it.

[Matlab plotcdf(0.7,12)]

# A reputation tarnished

Calculus question:

- What type of function has derivative 0?

# A reputation tarnished

Calculus question:

- What type of function has derivative 0?

Stroock's function has derivative 0 *almost everywhere*, but his function is not constant!

- "[Functions like mine] call into question the universal applicability of the Fundamental Theorem of calculus." —DWS
- "such functions are usually considered to be *pathological*. That is, most people suspect that these are not the sort of function on which they are likely to stumble unwittingly." —DWS

# A reputation tarnished

- "If nothing else, I hope that this note will

  **improve the reputation**

  of continuous, monotone, singular functions. Indeed, it will be shown that such functions arise inevitably whenever one deals with infinitely many tosses of an unfair coin."

Such crazy functions are not oddities.
They are a fact of life!

# Binary representation of a real number

*Exercise*. Represent $\frac{5}{8}$ in binary.

*Exercise*. Represent 1 in binary.

*Exercise*. Represent $\frac{1}{4}$ in binary.

# Definition of Stroock's function

Flip a coin.

$$X_1 \equiv \begin{cases} 1 & \text{if heads} \\ 0 & \text{if tails} \end{cases}$$

Repeat.

$$X_1, X_2, X_3, \ldots$$

Let $X_1, X_2, X_3, \ldots$ define a random real number,

$$Y \equiv \sum_{i=1}^{\infty} X_i 2^{-i}.$$

# Definition of Stroock's function

For example, if

$$(X_1, X_2, X_3, \dots) = (0, 1, 1, 0, 1, 0, 0, 0, 0, 0, \dots),$$

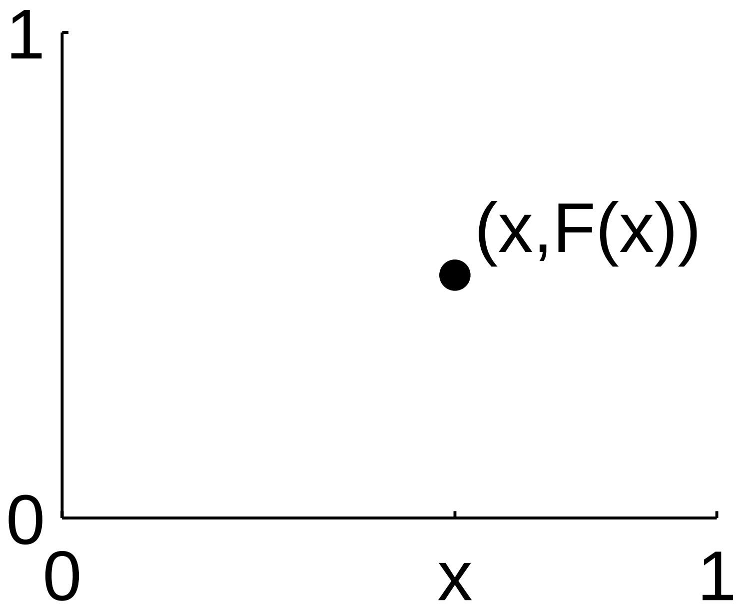
then

$$\begin{aligned} Y &= 0.\,0\,1\,1\,0\,1\,0\,0\,0\,0\,0\,\cdots_2 \\ &= \frac{1}{4} + \frac{1}{8} + \frac{1}{32} \\ &= 0.4062\dots. \end{aligned}$$

# Definition of Stroock's function

Let $F$ be the CDF of $Y$,

$$F(x) = P(Y \leq x).$$



# Definition of Stroock's function

Fact.

$$F(\tfrac{1}{2}) = P(Y \le \tfrac{1}{2}) = \tfrac{1}{2}.$$

*Proof.* What is $P(Y \le \frac{1}{2})$?

- Two cases produce $Y \le \frac{1}{2}$.
  - $Y = 0.0*****\cdots_2$, i.e., tails on the first toss (probability $\frac{1}{2}$)
  - $Y = 0.100000\cdots_2$, i.e., (probability 0).
- $P\left(Y \le \frac{1}{2}\right) = \frac{1}{2} + 0 = \frac{1}{2}$. □

There is a 50% chance that $Y \le \frac{1}{2}$.

# Definition of Stroock's function

Another fact.

$$F(\tfrac{1}{4}) = P(Y \leq \tfrac{1}{4}) = \tfrac{1}{4}.$$

*Proof.* What is $P(Y \leq \frac{1}{4})$?

- Two cases produce $Y \leq \frac{1}{4}$.
  - $Y = 0.00****\cdots_2$, i.e., tails-tails-$\cdots$ (probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$)
  - $Y = 0.010000\cdots_2$ (probability 0).
- $P\left(Y \leq \frac{1}{4}\right) = \frac{1}{2} \cdot \frac{1}{2} + 0 = \frac{1}{4}.$ □

There is a 25% chance that $Y \leq \frac{1}{4}$.

# Definition of Stroock's function

$$F(x) = P(Y \le x)$$

We have determined

$$F(\tfrac{1}{4}) = \tfrac{1}{4},$$
$$F(\tfrac{1}{2}) = \tfrac{1}{2},$$

which suggests

$$F(x) = x,$$

which is provably correct, but not terribly interesting.

[Matlab plotcdf(0.5,1)]

⋮

[Matlab plotcdf(0.5,10)]

# Definition of Stroock's function

*Now* things get exciting. Imagine that the coin is biased.

- probability of heads $p \neq \frac{1}{2}$
- probability of tails $q \equiv 1 - p$.

Let $F_p$ be the CDF of $Y$ when the coin has bias $p$.

$$F_p(x) = P(Y \leq x) \qquad \text{(remember: bias)}$$

# Definition of Stroock's function

<u>unbiased</u>

$$p = \tfrac{1}{2},\ q = \tfrac{1}{2}$$

$$X_i = 1 \text{ if heads},\ 0 \text{ if tails}$$

$$Y = \sum_{i=1}^{\infty} X_i 2^{-i}$$

<u>biased</u>

$$p \neq \tfrac{1}{2},\ q = 1 - p$$

$$X_i = 1 \text{ if heads},\ 0 \text{ if tails}$$

$$Y = \sum_{i=1}^{\infty} X_i 2^{-i}$$

$$P(Y \le \tfrac{1}{2}) = \tfrac{1}{2}$$

$$P(Y \le \tfrac{1}{2}) = q$$

$$P(Y \le \tfrac{1}{4}) = (\tfrac{1}{2})^2 = \tfrac{1}{4}$$

$$P(Y \le \tfrac{1}{4}) = q^2$$

$$P(Y \le \tfrac{3}{4}) = (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^2 = \frac{3}{4}$$

$$P(Y \le \tfrac{3}{4}) = q^2 + qp + pq = q^2 + 2pq$$

# Definition of Stroock's function

[Matlab plotcdf(0.7,2)]

⋮

[Matlab plotcdf(0.7,10)]

# Properties of Stroock's function

- non-constant
- monotone
- continuous
- derivative equals 0 *almost everywhere*
- starts at $(0,0)$, ends at $(1,1)$, arclength 2

# Properties of Stroock's function

- non-constant

# Properties of Stroock's function

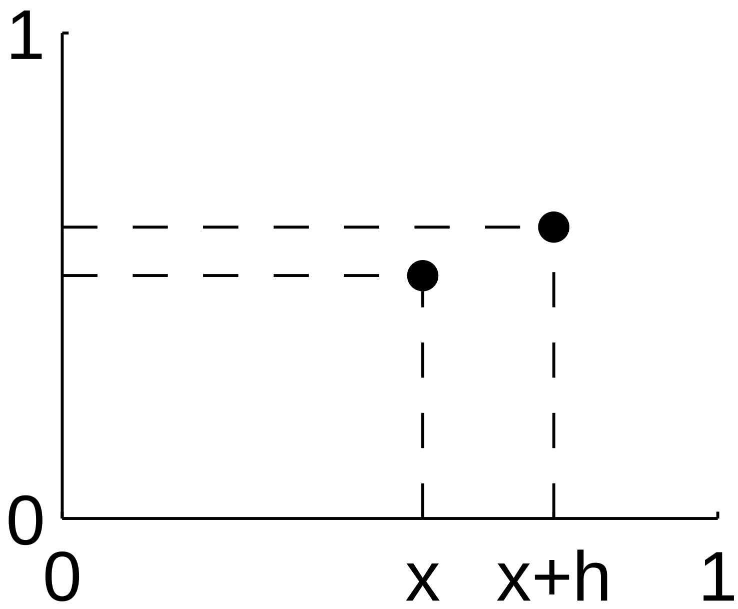
- monotonically increasing

If $h > 0$, then

$$F_p(x) \leq F_p(x+h),$$

i.e.,

$$F_p(x+h) - F_p(x) \geq 0.$$



# Properties of Stroock's function

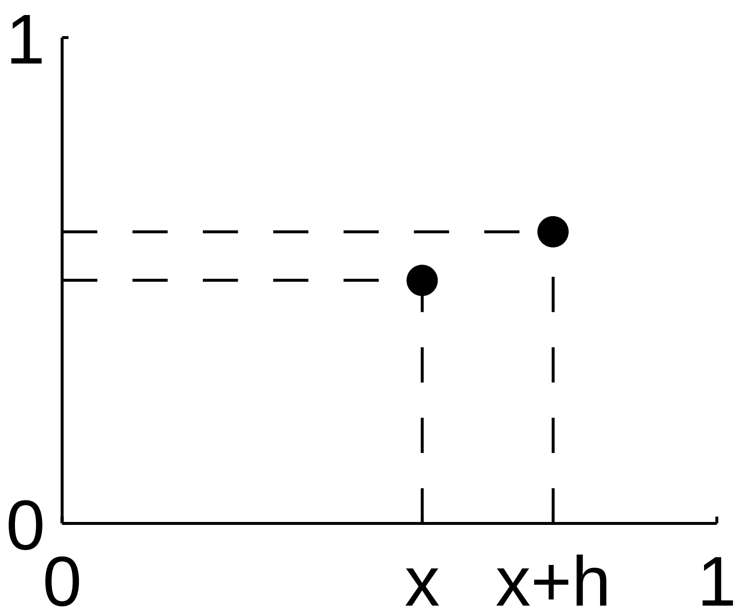
- continuous

As $h \to 0$ from the right,

$$F_p(x+h) \to F_p(x),$$

i.e.,

$$F_p(x+h) - F_p(x) \to 0.$$



# Properties of Stroock's function

- derivative equals 0 *almost everywhere*

*Vague idea of a partial proof.*

- The binary expansion of a "typical" $x$ has approximately half 0's and half 1's.
- For any $n$, sandwich $x$ in $L_n(x) < x \leq R_n(x)$, with $L_n(x)$ and $R_n(x)$ representable in $n$ bits.
- A difference quotient around $x$ is

$$\begin{aligned}\frac{F_p(R_n(x)) - F_p(L_n(x))}{R_n(x) - L_n(x)} &= \frac{P(L_n(x) < Y \leq R_n(x))}{2^{-n}} \\ &= 2^n P(\text{first } n \text{ bits of } Y \text{ and } L_n(x) \text{ agree}) \\ &\approx 2^n p^{n/2} q^{n/2} \\ &= (4pq)^{n/2}.\end{aligned}$$

- As long as $p \neq \frac{1}{2}$, we have $pq < \frac{1}{4}$, so $4pq < 1$. Hence, $(4pq)^{n/2} \to 0$ as $n \to +\infty$, i.e., as "$h \to 0$".

# Properties of Stroock's function

- starts at $(0, 0)$, ends at $(1, 1)$, arclength 2

We do not prove this, but here is what it means.

- Lay out a piece of string in the shape of the graph.
- Stretch the string into a straight line.
- The length of the string will be 2, the same as the stairstep length.

The surprising thing, of course, is that Stroock's function passes the vertical line test!

# References


- Stroock, Daniel W. "Doing analysis by tossing a coin." *Math. Intelligencer* 22 (2000), 66–72.